Eudoxus, Euclid and Hölder on Measurement, Ratio and Proportion

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The Klein Project, at MathFest 2010, Pittsburgh

The One-Slide Version of This Talk

- The terms A and B of a Euclidean ratio $A:B$ need not be numbers. They are things (commonly called "magnitudes") that have a measurable attribute in common.
- $A:B$ is a real number; it is the measure of A by B.
- Ratio, in this sense, underlies measurement in the physical sciences.
- Keeping this perspective on measurement in thoughtful view throughout school math can support intellectual coherence.

Plan

1. Motivate a discussion of measurement, ratio and proportion from within the traditional treatments of these topics in school math.
2. Sketch the theory ratio and proportion presented by Euclid *Elements*, Book V, with emphasis on its relevance to theory of measurement.
3. Link this advanced perspective to themes in the *Common Core State Standards* and to other high-level issues in curriculum design.

Outline

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Proportional Reasoning

Ratio and proportion have always been a part of school mathematics. Problems similar to those collected by Fibonaci in the *Liber Abaci* (1202) make up much of what is today called “proportional reasoning.”

Fibonacci presented solutions by means of his “Principal Method,” a version of the even more ancient “Rule of Three.” Students still learn practice similar procedures in many middle-school curricula.

Measure Numbers in *Liber Abaci*

Opening of Ch. 8, translated and paraphrased

Four numbers are found in these negotiations, of which three are known and one is unknown. The first is the number of items sold, or the weight or measure of the sale: a hundred hides or goatskins, or a hundredweight or a hundredpound, or pounds, or ounces, or pints of oil, or sestarios of corn, or bundles of cloth. The second is the price of the sale: a quantity of denari, or of bezants, or of tareni . . . The third is another quantity of the same merchandise, and the fourth is the unknown price.

The Learner's Perspective

We reached our new home [in Spencer County, Indiana] in my eighth year [1817]... It was a wild region, with many bears and other wild animals, still in the woods. There I grew up. There were some schools, so called; but no qualification was ever required of a teacher beyond 'readin, writin, and cipherin' to the Rule of Three.

Abraham Lincoln (brief autobiography of 1860)

In the surviving pages of Lincoln's school notebook, one can find passages copied from Thomas Dilworth's *Schoolmaster's Assistant*, as well as a solution to the following problem from that book: *If 3 ounces of silver cost 17 shillings what will 48 ounces cost?*

Outline

The Mathematician's Perspective 1

The following passages are from texts or papers by mathematicians attempting to clarify the meaning of ratio and proportion in school math. Because I am quoting out of context, I will omit the authors' names.

- Ratios are essentially just fractions, and understanding and working with ratios and proportions really just involves understanding and working with multiplication, division, and fractions. . . . To say that two quantities are in a ratio A to B means that for every A units of the first quantity there are B units of the second quantity.

The Mathematician's Perspective 2

- By definition, given two . . . [numbers] A and B, where $B \neq 0$ and both refer to the same unit (i.e., they are points on the same number line), the ratio of A to B, sometimes denoted by $A : B$, is the . . . [number] A/B.
- The ratio of two quantities of the same kind is the quotient of their measures. . . . An equality of two ratios is called a proportion. (*The Van Nostrand Reinhold Concise Encyclopedia of Mathematics.* 1977. Page 38.)

The Mathematician's Perspective 3

- We say that the ratio between two quantities is $A:B$ if there is a unit so that the first quantity measures A units and the second measures B units. . . . Two ratios are equivalent if one is obtained from the other by multiplying or dividing all the measurements by the same nonzero number. . . . A proportion is a statement that two ratios are equal.

Outline

1. Background and Motivation
   - Traditions in School Math
   - Mathematicians Speak
   - Summary
2. Euclid, *Elements*, Book V
   - Introduction
   - What Magnitudes Are
   - Euclidean Ratios are Numbers
3. Measurement, ratio and proportion in CCSSM
   - Magnitudes and Kinds
   - Units and the Measurement Process
   - Conversions and Dimensional Analysis

Numbers and Things

Ratio is not merely division of numbers. References to units and measurement appear in all the passages quoted. All the commentators acknowledge conceptual issues at the interface between:

- things in the world that are not numbers, and
- our representations of the relations between things by means of numbers.

Outline

Influence of Euclid on School Math

Book V of Euclid's *Elements* presents a theory (attributed by commentators to Eudoxus) of ratio and proportion for geometric magnitudes. This theory permits the formation of ratios of incommensurables.

Book V has had numerous superficial influences on school math:

- The terms of a ratio must be of the same kind
- A proportion is an equality of ratios

(Of course, Book V has deep connections to mathematics, e.g., as a precursor of Dedekind's theory of real numbers.)

The Key Definitions

Definition 3. A ratio is a sort of relation in respect of size between two magnitudes of the same kind.

Definition 5. Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Outline

Number and Quantity

Euclidean magnitudes are

- *not* numbers,
- but **things** (in the world or in thought)
- that have measurable attributes.

Surely, this is something we should have learned in Kindergarten.

Some day, perhaps Americans in at least 48 states will . . .

Common Core State Standards

Mathematics / Kindergarten / Measurement and Data

Describe and compare measurable attributes.

1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
2. Directly compare two objects with a measurable attribute in common, to see which object has “more of”/“less of” the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

What properties do Euclidean magnitudes have?

1) Magnitudes are of several different kinds, *e.g.*, segments, polygonal regions, volumes, angles, weights, durations, etc.

2) Given two magnitudes **of the same kind**, exactly one of the following is true: a) they are the same with respect to size, b) the first exceeds the second or c) the second exceeds the first.

What properties do Euclidean magnitudes have?

3) Magnitudes **of the same kind** may be added to one another—or a given magnitude may be added to itself one or more times—to yield a new magnitude of the same kind that is larger than any summand. No matter how the addition is performed, the outcome has the same size. *Furthermore*, given two magnitudes of the same kind but of different size, a part of the larger equivalent to the smaller may be removed, and no matter how this removal is done, the remainders are equivalent.

What properties do Euclidean magnitudes have?

4) The relationships of equivalence and of excess are compatible with addition and subtraction in the sense that if equivalent magnitudes are added to (or taken from) each of two others, the resulting magnitudes will be in the same relation as the originals.

Euclidean Magnitudes: Summary

The Euclidean magnitudes of any given kind form a sub-semigroup of an archimedean totally-ordered group.

Outline

What does Definition 5 mean?

Suppose A and U are magnitudes of the same kind. Then the ratio $A : U$ is completely characterized by the sets:

$$\{\,(m,n) \in \mathbb{N} \times \mathbb{N} \mid mU < nA\,\},$$

and

$$\{\,(m,n) \in \mathbb{N} \times \mathbb{N} \mid mU = nA\,\}.$$

(This is a direct, literal rephrasing of Definition 5.)

Definition 5 elaborated (via Dedekind)

This set, in turn, is completely determined by the bounded lower set:

$$\{\, m/n \in \mathbb{Q} \mid mU < nA \,\},$$

and hence determines a real number:

$$[\, A : U \,] := \sup\{\, m/n \in \mathbb{Q} \mid mU < nA \,\}.$$

$[\, A : U \,]$ may be called, "A measured by U."

Hölder's Theorem (1901)

Theorem. Every archimedean totally-ordered group G is order-isomorphic to a sub-group of the additive real numbers.

Sketch of Proof. Select $u \in G$. Then

- $a \mapsto [\,a\!:\!u\,] : G \to \mathbb{R}$ is injective and order-preserving, and
- $[\,a+b\!:\!u\,] = [\,a\!:\!u\,] + [\,b\!:\!u\,]$

The proof of the second part is similar in form to the proof of continuity of addition.

Otto Hölder. (1901). Die Axiome der Quantität und die Lehre vom Mass. (English transl. in *J. Math. Psychology* 40 (1996).)

Research Connection

Hölder's Theorem has the following "globalizaion":

Localic Yosida Representation. Any archimedean lattice-ordered group with weak order unit is order-isomorphic to an ℓ-group of real-valued functions on a regular Lindelöf locale. [*Trans. Am. Math. Soc.* 331 (1992), 265–279.]

The Change of Unit and Multiplication Theorems

Theorem (Change of Unit). Suppose G is an archimedean totally-ordered group and $a, u, w \in G$. Then

$$[a : w] = [a : u][u : w].$$

Theorem. Suppose G, H and K are archimedean totally-ordered groups and $(g, h) \mapsto g \cdot h : G \times H \to K$ is an order-homomorphism in each variable separately. Then

$$[a \cdot b : u \cdot v] = [a : u][b : v].$$

Example

Problem. It's 2.2 miles from my house to school. What's that in kilometers?

Solution 1.

$$2.2 \text{ miles} \times \frac{1.61 \text{ kilometers}}{1 \text{ mile}} = 3.54 \text{ kilometers}$$

Solution 2.

[distance : mile]	[mile : kilom.]	=	[distance : kilom.]	
2.2	$\times$ 1.61	=	3.54	

Which is easier to understand?

Outline

Magnitudes and Kinds

CCSSM progressively introduces different kinds through several grades, with attention to the order and additive structures within in each kind

- Grade 1: Length
- Grade 2: Area
- Grade 3: Time, Liquid measure, Mass
- Grade 4: Angle
- Grade 5: Volume

A curriculum that recognizes and highlights the analogies between kinds—the recurring order and additive structures—will promote coherence.

Outline

Units and the Measurement Process

- Grade 1: “Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end . . .”
- Grade 3: “Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).”
- Grade 4: "An angle that turns through n one-degree angles is said to have an angle measure of n degrees.”
- Grade 5: “A cube with side length 1 unit, called a ‘unit cube,’ is said to have ‘one cubic unit’ of volume, and can be used to measure volume. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.”

Outline

Conversions and Dimensional Analysis

- Grade 2: Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
- Grade 4: “Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit.”
- Grade 6: “Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.”
- HS (Overview): “In real world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. . . . ”

Some references

- Otto Hölder (1901) Die Axiome der Quantität und die Lehre vom Mass. Translated in *J. Math. Psychology* 40 (1996).
- Joel Michell (1999), *Measurement in Psychology.* Cambridge U. Pr.
- D. Krantz, R. Luce, P. Suppes & A. Tversky. *Foundations of Measurement*. (In three volumes, 1971–1990)